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(Fermionic) mass meets (intrinsic) curvature

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Abstract

Using the notion of vacuum pairs we show how the (square of the) mass matrix of the fermions can be considered geometrically as curvature. This curvature together with the curvature of space–time, defines the total curvature of the Clifford module bundle representing a "free" fermion within the geometrical setup of spontaneously broken Yang–Mills–Higgs gauge theories. The geometrical frame discussed here gives rise to a natural class of Lagrangian densities. It is shown that the geometry of the Clifford module bundle representing a free fermion is described by a canonical spectral invariant Lagrangian density.

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1. Introduction

In a recent paper we showed how bosonic mass is related to the extrinsic geometry of a chosen vacuum (cf. [8]). In the present paper we will show how the mass of a fermion is related to the curvature of the Hermitian vector bundle that represents the (free) fermion in question. The geometrical context we work with is that of Clifford module bundles and operators of Dirac type. Using the notion of vacuum pairs we will show how the fermionic mass matrix permits decomposition of the fermion bundle into the Whitney sum of certain Hermitian (line) bundles representing (almost) free fermions of specific mass. A natural class of non-flat connection exists on this type of bundle which is defined by the space–time metric together with the mass of the fermion. The corresponding Dirac operator $\not{\partial}_{\mathcal{D}}$ is the geometrical analogue of Dirac's first-order operator $i\not{\partial} - m$ that has been introduced

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to relativistically describe the dynamics of a free fermion of mass m. We show how a (linear) fluctuation of the vacuum yields the Dirac-type operator D_{Y} usually referred to as Dirac–Yukawa operator.

The basic question addressed in this paper is how to understand the notion of fermionic mass from a geometrical perspective. Interestingly, this question is tied to several other basic questions like, for instance, how to understand Dirac's famous first-order operator as a "true" Dirac operator. Or, since the notion of mass is related to the notion of a "free" particle (i.e. to a dynamically closed system[1]), how one can understand the notion of "freeness" within the geometrical setup of Yang–Mills gauge theories. Another question along this line of thought is how one can geometrically understand what is usually referred to as "particle multiplet". Usually, elementary particles are described by ("quantized") fields Ψ that are defined on space–time $\mathcal{M}$ without referring to a geometrical description of the particles themselves. Moreover, it is assumed that some of these fields actually constitute a "fermion multiplet" with respect to some "internal symmetry group" G. As is well-known, for instance, in the case of the Standard Model of particle physics the fields representing a (left-handed) electron and a (left-handed) neutrino together build a (left-handed) fermion doublet with respect to the symmetry group $\mathrm{SU}(2) \times \mathrm{U}(1)$ of the electroweak interaction. Thus, if we believe in the Standard Model, neither an electron nor a neutrino itself can actually be regarded as a fundamental particle. Moreover, electromagnetism itself turns out to be an effective interaction only. Consequently, the field Ψ describing the fermion doublet, e.g., of an electron and a neutrino is considered to decompose into $\Psi = (\Psi_1, \Psi_2)$. The gauge symmetry of the electroweak interaction then manifests it selves in the arbitrariness of which component of Ψ is identified with the field describing, e.g., an electron. In other words, one usually has to choose a gauge in order to identify, for instance, Ψ_1 with the electron field. However, such a description seems to be unsatisfying since on the one hand the choice of a gauge is a purely mathematical operation (i.e. it cannot be achieved experimentally). On the other hand, there is no doubt that an electron exists in nature as an object of its own. It thus cannot depend on some choice of gauge.

From a purely mathematical point of view the space where Ψ takes its values forms a specific representation of the symmetry group G in question. For instance, in the case of the electroweak group this space[2] is identified with $\mathbb{C}^2$. Then, the fermionic mass matrix M_{F} provides a natural decomposition of this space into the eigenspaces of the mass matrix. In the case of $G = \mathrm{SU}(2) \times \mathrm{U}(1)$ one then obtains

$$\mathbb{C}^2 \simeq W_{\text{electron}} \oplus W_{\text{neutrino}}. \tag{1}$$

This decomposition breaks the original gauge symmetry since the fermionic mass matrix M_{F} does not, in general, lie in the commutant of the symmetry group G. However, the point is that the decomposition (1) does not refer to any specific gauge. The decomposition is "natural" with respect to the additional piece of input that comes from the fermionic mass matrix. However, in order to put the decomposition (1) in an appropriate geometrical context

[1] It is well-known that, e.g., in the context of the strong interaction there is no unique definition of mass of the quarks.

[2] Here, only one generation of left-handed fermions is taken into account in order to simplify the discussion. The general case is discussed hereafter.

without assuming the triviality of the underlying gauge bundle we first have to globalize the fermionic mass matrix. This will be done by using vacuum pairs similar to the case of the bosonic mass matrices (see loc sit). As a consequence we will see that the mass matrix has a simple geometrical interpretation in terms of curvature and that Dirac's operator can in fact be considered as a Dirac-type operator.

2. Orbit bundles and vacuum pairs

To get started we first summarize the notion of vacuum pairs that has been introduced in [8]. For this let $\mathcal{P}(\mathcal{M}, G)$ be a smooth principal G-bundle $P \stackrel{\pi_P}{\to} \mathcal{M}$ over a smooth orientable (pseudo)Riemannian (spin-) manifold $(\mathcal{M}, g_{\mathrm{M}})$ of dimension $\dim(\mathcal{M}) = 2n$. Here, G is a semi-simple compact real Lie group with Lie algebra Lie(G). The corresponding gauge group is denoted by $\mathcal{G}$. Let $G \stackrel{\rho_{\mathrm{H}}}{\to} \mathrm{Aut}(\mathbb{C}^{N_{\mathrm{H}}})$ be a unitary representation of G. Also, let $\mathbb{C}^{N_{\mathrm{H}}} \stackrel{V_{\mathrm{H}}}{\to} \mathbb{R}$ be a smooth G-invariant function that is bounded from below. Moreover, it is assumed that its Hessian is positive definite transversally to the orbits of minima. We call V_{H} a *general Higgs potential*. The triple $(\mathcal{P}(\mathcal{M}, G), \rho_{\mathrm{H}}, V_{\mathrm{H}})$ defines the geometrical data of a *Yang–Mills–Higgs gauge theory*. We call the Hermitian vector bundle ξ_{H}:

$$\pi_{\mathrm{H}}\colon E_{\mathrm{H}} =: P \times_{\rho_{\mathrm{H}}} \mathbb{C}^{N_{\mathrm{H}}} \to \mathcal{M}, \qquad \mathfrak{z} = [(p, \mathbf{z})] \mapsto \pi(p), \tag{2}$$

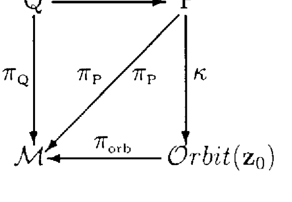
the *Higgs bundle* with respect to the above given data. It is assumed to geometrically represent the Higgs boson. Correspondingly, a state of the Higgs boson is geometrically represented by a section of the Higgs bundle.

Each minimum $\mathbf{z}_0 \in \mathbb{C}^{N_{\mathrm{H}}}$ defines a sub-bundle of the Higgs bundle. For this, let orbit$(\mathbf{z}_0) \subset \mathbb{C}^{N_{\mathrm{H}}}$ and $I(\mathbf{z}_0) \subset G$ be the orbit and the isotropie group of the minimum. We call the fiber bundle $\xi_{\mathrm{orbit}(\mathbf{z}_0)}$:

$$\pi_{\mathrm{orb}}\colon \mathcal{O}rbit(\mathbf{z}_0) =: P \times_{\rho_{\mathrm{orb}}} \mathrm{orbit}(\mathbf{z}_0) \to \mathcal{M} \tag{3}$$

the *orbit bundle* with respect to the data defining a Yang–Mills–Higgs gauge theory (see above). Here, $\rho_{\mathrm{orb}} =: \rho_{\mathrm{H}}|_{\mathrm{orbit}(\mathbf{z}_0)}$.

Notice that, since $\xi_{\mathrm{orbit}(\mathbf{z}_0)} \subset \xi_{\mathrm{H}}$, every section $\mathcal{V} \in \Gamma(\xi_{\mathrm{orbit}(\mathbf{z}_0)})$ of the orbit bundle can be also considered as a section of the Higgs bundle. There is a one-to-one correspondence between the sections $\mathcal{V}$ and "H-reductions" of $\mathcal{P}(\mathcal{M}, G)$, where $H \simeq I(\mathbf{z}_0)$. More precisely, let H be the unique subgroup of G that is similar to the isotropie group of the minimum $\mathbf{z}_0$. Then, every section $\mathcal{V} \in \xi_{\mathrm{orbit}(\mathbf{z}_0)}$ uniquely corresponds to a principal H-bundle $\mathcal{Q}(\mathcal{M}, H)$ together with an embedding $Q \stackrel{\iota}{\to} P$, such that the following diagram commutes (see, for instance, Chapter 1, Proposition 5.6 in [5])



Note that $P \stackrel{\kappa}{\rightarrow} \mathcal{O}rbit(\mathbf{z}_0)$ is a principal H-bundle, where $\kappa(pg) =: [(p, \rho(g)\mathbf{z}_0)]$ denotes the canonical projection.

We call a section $\mathcal{V}$ of the orbit bundle a *vacuum section* and $(\mathcal{Q}, \iota)$ the corresponding *vacuum* with respect to the minimum $\mathbf{z}_0$. We denote by $\mathcal{H}$ the gauge group that is defined by the vacuum $(\mathcal{Q}, \iota)$ and call it the *invariance group* of the vacuum. A Yang–Mills–Higgs gauge theory is called *spontaneously broken* by a vacuum $(\mathcal{Q}, \iota)$ if the invariance group of the latter is a proper subgroup of the original gauge group $\mathcal{G}$. The gauge theory is called *completely broken* by the vacuum if the appropriate invariance group is trivial. We call a vacuum $(\mathcal{Q}, \iota)$ trivial if $\mathcal{Q}(\mathcal{M}, \mathrm{H})$ is equivalent to the trivial principal H-bundle $\mathcal{M} \times H \stackrel{\mathrm{pr}_1}{\rightarrow} \mathcal{M}$. Notice that even a trivial gauge bundle $\mathcal{P}(\mathcal{M}, G)$ may have nontrivial vacua.

A connection A on $\mathcal{P}(\mathcal{M}, G)$ is called *reducible* with respect to a given vacuum $(\mathcal{Q}, \iota)$ (or *compatible* with respect to the vacuum section $\mathcal{V}$) if $\iota^* A$ is a connection on $\mathcal{Q}(\mathcal{M}, \mathrm{H})$. Let, respectively, $\mathcal{A}(\xi_\mathrm{H})$ and $\Gamma(\xi_\mathrm{H})$ be the affine set of all associated connections on the Higgs bundle and the module of sections of the Higgs bundle. A Yang–Mills–Higgs pair $(\Theta, \mathcal{V}) \in \mathcal{A}(\xi_\mathrm{H}) \times \Gamma(\xi_\mathrm{H})$ is called a *vacuum pair* if $\mathcal{V}$ is a vacuum section and Θ corresponds to a flat connection on $\mathcal{P}(\mathcal{M}, G)$ that is compatible with the vacuum section. Clearly, a vacuum pair minimizes the energy functional that corresponds to the Yang–Mills–Higgs action with respect to the data $(\mathcal{P}(\mathcal{M}, G), \rho_\mathrm{H}, V_\mathrm{H})$. In particular, a vacuum section $\mathcal{V}$ corresponds to a ground state of the Higgs boson. A vacuum pair $(\Theta, \mathcal{V})$ geometrically generalizes the canonical vacuum pair $(d, \mathbf{z}_0)$ in the case of the trivial gauge bundle $\mathcal{M} \times G \stackrel{\mathrm{pr}_1}{\rightarrow} \mathcal{M}$. In fact, it can be shown that in the case of a simply connected space–time there is at most one vacuum pair to the orbit of a given minimum $\mathbf{z}_0$ apart from gauge equivalence. Moreover, this pair is gauge equivalent to the canonical vacuum pair.[3] In general, however, the data $(\mathcal{P}(\mathcal{M}, G), \rho_\mathrm{H}, V_\mathrm{H})$ may give rise to gauge inequivalent vacua even in the case of only one nontrivial orbit of minima.

We have summarized the basic geometrical notion that we need to globalize the fermionic mass matrix. This will be discussed in the next section.

3. Clifford module bundles and the fermionic mass matrix

Having chosen a spin structure $\mathcal{S}$ we denote the appropriate spinor bundle by ξ_S. Let $G \stackrel{\rho_\mathrm{F}}{\rightarrow} \mathrm{Aut}(\mathbb{C}^{N_\mathrm{F}})$ denote a second unitary representation of G. The corresponding associated Hermitian vector bundle ζ_F is defined by

$$\pi_\mathrm{F} : E_\mathrm{F} =: P \times_{\rho_\mathrm{F}} \mathbb{C}^{N_\mathrm{F}} \rightarrow \mathcal{M}. \tag{4}$$

We then call the twisted spinor bundle

$$\xi_\mathrm{F} =: \xi_\mathrm{S} \otimes \zeta_\mathrm{F} \tag{5}$$

[3] Here, a minimum $\mathbf{z}_0$ is considered as the vacuum section $\mathcal{M} \stackrel{\mathbf{z}_0}{\rightarrow} \mathcal{M} \times \mathrm{orbit}(\mathbf{z}_0)$, $x \mapsto (x, \mathbf{z}_0)$ and d is the covariant derivative with respect to the trivial connection. Indeed, in physics the mechanism of spontaneous symmetry breaking refers to the canonical vacuum pair $(d, \mathbf{z}_0)$. This is consistent for in particle physics the common model of space–time is that of $(\mathcal{M}, g_\mathrm{M}) \simeq \mathbb{R}^{1,3}$.

the *fermion bundle* with respect to the data $(\mathcal{P}(\mathcal{M}, G), \rho_F, \mathcal{S})$. It geometrically represents a particle of spin one-half. In what follows we will assume that the fermion bundle is also $\mathbb{Z}_2$-graded with respect to the "inner degrees of freedom", i.e. $\zeta_F = \zeta_{F,L} \oplus \zeta_{F,R}$.

On such a $\mathbb{Z}_2$-graded fermion bundle there exists a distinguished class of first-order differential operators called *Dirac operators of simple type* (cf. [1,6]). More precisely, let ξ_{Cl} be the Clifford bundle with respect to $(\mathcal{M}, g_M)$. The fermion bundle forms a natural left module of the Clifford bundle. The corresponding action is denoted by γ. By an operator of Dirac type we mean any odd first-order differential operator D acting on the module of sections $\Gamma(\xi_F)$ such that D^2 is a generalized Laplacian, i.e. $[D, [D, f]] = \pm 2g_M(df, df)$ for all $f \in \mathcal{C}^\infty(\mathcal{M})$ (see, e.g., Chapter 3.3 in [2]). Let $\mathcal{D}(\xi_F)$ be the affine set of all operators of Dirac type which are compatible with the Clifford action γ, i.e. $[D, f] = \gamma(df)$. The appropriate vector space is given by $\Omega^0(\mathcal{M}, \mathrm{End}^-(\mathcal{E}))$, where $\mathcal{E} =: \mathrm{S} \otimes E_F$ is the total space of the fermion bundle. We also denote by $\mathcal{A}(\xi_F)$ the affine set of all (associated) connections on ξ_F. The corresponding vector space is given by $\Omega^1(\mathcal{M}, \mathrm{End}^+(\mathcal{E}))$. In general, one has $\mathcal{D}(\xi_F) \simeq \mathcal{A}(\xi_F)/\mathrm{Ker}(\gamma)$. Thus, there is a whole class $[A]$ of connections on the fermion bundle corresponding to each Dirac type operator D. However, there is a distinguished class of connections on the fermion bundle that is constructed as follows: an operator $D \in \mathcal{D}(\xi_F)$ is called of simple type if its Bochner–Laplacian Δ_D is defined by a Clifford connection $A \in \mathcal{A}_{Cl}(\xi_F) \subset \mathcal{A}(\xi_F)$. Here, $\mathcal{A}_{Cl}(\xi_F)$ denotes the affine subset of Clifford connections on the fermion bundle. They are characterized by the covariant derivatives ∂_A that fulfil $[\partial_{A,X}, \gamma(a)] = \gamma(\partial^{Cl}_X a)$ for all sections $a \in \Gamma(\xi_{Cl})$ and tangent vector fields $X \in \Gamma(\tau_M)$. Here, ∂^{Cl} is the covariant derivative that is defined by the canonical connection on the Clifford bundle ξ_{Cl}. In the case of a twisted spinor bundle Clifford connections are tensor product connections and thus are parameterized by Yang–Mills connections on ζ_F. It can be shown that D is of simple type iff it reads (cf. [1,6])

$$D \equiv \partial\!\!\!/_{A,\phi} = \partial\!\!\!/_A + \gamma_5 \otimes \phi, \tag{6}$$

where γ_5 is the grading operator on ξ_S and $\phi \in \Omega^0(\mathcal{M}, \mathrm{End}^-(E_F))$.

Of course, any twisted Spin-Dirac operator $\partial\!\!\!/_A =: \gamma \circ \partial_A$ is of simple type. However, the most general Dirac operator of simple type on the fermion bundle (more general: on any "twisted" Clifford module bundle) is given by (6). Notice that these more general Dirac operators exist only if ζ_F is $\mathbb{Z}_2$-graded. The connection class of a Dirac operator of simple type has a natural representative. The corresponding covariant derivative reads

$$\partial_{A,\phi} = \partial_A + \xi \wedge (\gamma_5 \otimes \phi). \tag{7}$$

Here, $\xi \in \Omega^1(\mathcal{M}, \mathrm{End}^-(\mathcal{E}))$ is the canonical one form that fulfils the following criteria: (a) it is covariantly constant with respect to every Clifford connection; (b) it defines a right inverse of the Clifford action γ (cf. [6]).

Definition 3.1. Let $(\mathcal{P}(\mathcal{M}, G), \rho_H, V_H)$ be the data of a Yang–Mills–Higgs gauge theory and let ξ_F be the fermion bundle with respect to $(\mathcal{P}(\mathcal{M}, G), \rho_F, \mathcal{S})$. A linear mapping

$$G_Y : \Gamma(\xi_H) \to \Gamma(\xi_{\mathrm{End}^- E_F}), \qquad \varphi \mapsto \phi_Y =: G_Y(\varphi), \tag{8}$$

such that $G_{\mathrm{Y}}(\varphi)^{\dagger} = -G_{\mathrm{Y}}(\varphi)$ is called a "Yukawa mapping". A Dirac operator of simple type

$$D_{\mathrm{Y}} =: \not{\partial}_{\mathrm{A}} + \gamma_5 \otimes \phi_{\mathrm{Y}} \tag{9}$$

is called a "general Dirac–Yukawa operator". Moreover, if $(\mathcal{Q}, \iota)$ is a vacuum that spontaneously breaks the gauge symmetry, then the Hermitian section

$$-\mathrm{i}\mathcal{D} =: -\mathrm{i}G_{\mathrm{Y}}(\mathcal{V}) \tag{10}$$

$$-\mathrm{i}\mathcal{D} \equiv \begin{pmatrix} 0 & M_{\mathrm{F}} \\ M_{\mathrm{F}}^{\dagger} & 0 \end{pmatrix} \tag{11}$$

is called the "fermionic mass matrix".

Clearly, a necessary condition for the existence of a Yukawa mapping is that the representation ρ_{H} and the fermionic representation ρ_{F} are not independent of each other. For instance, in the case of the (minimal) Standard Model the existence of (8) is equivalent to the validity of the well-known relations between the "hyper-charges" of the leptons, the quarks and the Higgs boson (see, e.g. [6]). The constants which parameterize the mapping G_{Y} are usually referred to as "Yukawa coupling constants".

In this section we have seen how the notion of vacuum (pairs) can be used to consider the fermionic mass matrix as a globally defined (odd) operator acting on the states of a fermion that is geometrically defined by the data $(\mathcal{P}(\mathcal{M}, G), \rho_{\mathrm{F}}, \mathcal{S})$. In the next section we will show how the fermionic mass matrix $\mathcal{D}$ (together with the (pseudo) metric g_{M}) defines a canonical connection on the "reduced" fermion bundle. A necessary condition for this connection to be flat is that the (almost) "free fermions" are massless. Moreover, the fermionic mass matrix will provide us with a geometrical interpretation of the "minimal coupling" in terms of the physically intuitive notion of "fluctuating vacua". The main feature of this geometrical interpretation is that, besides the Yang–Mills boson, the minimal coupling naturally includes the gravitational field and the Higgs boson.

4. Dirac–Yukawa operators as fluctuating vacua

The Yukawa mapping (8) permits us to consider a section of the Higgs bundle (i.e. a state of the Higgs boson) as an (odd) endomorphism acting on the fermion bundle. In particular, a vacuum pair $(\Theta, \mathcal{V})$ defines a Dirac–Yukawa operator

$$\not{\partial}_{\mathcal{D}} =: \not{\partial} + \gamma_5 \otimes \mathcal{D} \tag{12}$$

acting on sections of the *reduced fermion bundle* $\xi_{\mathrm{F,red}} =: \xi_{\mathrm{S}} \otimes \zeta_{\mathrm{F,red}}$, with $\zeta_{\mathrm{F,red}}$ defined by

$$\pi_{\mathrm{F,red}} : E_{\mathrm{F,red}} =: \mathcal{Q} \times_{\rho_{\mathrm{F}},\mathrm{red}} \mathbb{C}^{N_{\mathrm{F}}} \to \mathcal{M}, \qquad \mathfrak{z} = [(q, \mathbf{z})] \mapsto \pi_{\mathcal{Q}}(q). \tag{13}$$

Here, $\rho_{F,red} =: \rho_F|_H$. Notice that $\xi_{F,red} \simeq \xi_F$. Thus, a section of the reduced fermion bundle can be considered as a state of the fermion that refers to a particular vacuum (pair).[4]

As a consequence, a vacuum pair defines a natural non-flat connection on the reduced fermion bundle. This connection is defined by the *Dirac–Yukawa operator in the vacuum state* (12). The appropriate covariant derivative reads

$$\partial_{\mathcal{D}} =: \partial + \xi \wedge (\gamma_5 \otimes \mathcal{D}). \tag{14}$$

The (total) curvature on $\xi_{F,red}$, which is defined by $(g_M, \Theta, \mathcal{V})$, is given by[5]

$$\mathcal{F}_{\mathcal{D}} = \not{\mathcal{R}} + m_F^2 \xi \wedge \xi, \tag{15}$$

where $\not{\mathcal{R}}$ denotes the lifted (pseudo)Riemannian curvature tensor with respect to g_M and $i m_F =: \gamma_5 \otimes \mathcal{D}$.

The relative curvature $\mathcal{F}_{\mathcal{D}}^{\mathcal{E}/S} = m_F^2\, \xi \wedge \xi$ on the reduced fermion bundle is thus defined by the (square of the) mass matrix of the fermion with respect to the chosen vacuum (pair). Like in the case of the bosonic mass matrices we have the following lemma.

Lemma 4.1. *The spectrum of the fermionic mass matrix is constant and only depends on the orbit of the minimum* $\mathbf{z}_0$ *of a general Higgs potential. Moreover, the mass matrix lies within the commutant of the invariance group of the vacuum chosen. Hence, the reduced fermion bundle splits into the Whitney sum of the eigenbundles of the fermionic mass matrix, i.e.*

$$\xi_{F,red} = \bigoplus_{m^2 \in \mathrm{spec}(m_F^2)} \xi_{F(m^2)}, \tag{16}$$

where $\xi_{F(m^2)} =: \xi_{F,L(m^2)} \oplus \xi_{F,R(m^2)}$.

Proof. The argument is very much the same as in the case of the bosonic mass matrices. It relies on the fact that, independently of the vacuum $(\mathcal{Q}, \iota)$, the corresponding vacuum section reads $\mathcal{V}(x) = [(\iota(q), \mathbf{z}_0)]|_{q \in \pi_Q^{-1}(x)}$. Thus, the spectrum of the fermionic mass matrix is independent of $x \in \mathcal{M}$. Of course, if two minima $\mathbf{z}_0, \mathbf{z}'_0$ of a given general Higgs potential V_H are on the same orbit, then the corresponding vacua are equivalent. As a result, the spectrum of m_F^2 only depends on the orbit of some minimum. By the very construction of the fermionic mass matrix we have $[\mathcal{D}, \rho_F(h)] = 0$ for all $h \in \mathcal{H}$, such that $H \simeq I(\mathbf{z}_0)$. Since $m_F \in \Omega^0(\mathcal{M}, \mathrm{End}^-(\mathcal{E}))$ is constant on the reduced fermion bundle one can decompose the latter with respect to the eigenbundles of $M_F^\dagger M_F$ and $M_F M_F^\dagger$. □

For fixed $m^2 \in \mathrm{spec}(m_F^2)$ the Clifford module bundle $\xi_{F(m^2)}$ is regarded as the geometrical analogue of an *almost free fermion of mass m*. Here, “almost” refers to the circumstance

4 This is analogous to the reduced tangent bundle of an $O(2n)$-reduction of the frame bundle of $\mathcal{M}$: a local frame corresponds to $2n$ locally linear independent sections of the tangent bundle τ_M that are orthonormal with respect to the chosen reduction.

5 We would like to point out that all of this can also be defined without assuming the existence of a spin structure. Thus, it is the gravitational field together with the vacuum (pair) that counts and not so much the spin structure $\mathcal{S}$. At least, this holds true as long as the notion of anti-particles is not taken into account.

that neither the connection Θ, nor the reduced representation $\rho_{\mathrm{F,red}}$ is trivial, in general. Therefore, a non-trivial vacuum together with the topology of space–time may give rise to a non-trivial holonomy group analogously to the well-known Aharonov–Bohm effect. Notice that, if the spectrum of the fermionic mass matrix is non-degenerated, then $\xi_{\mathrm{F,red}}$ decomposes into the Whitney sum of the tensor product of the spinor bundle and appropriate Hermitian line bundles.

In what follows we will rewrite a general Dirac–Yukawa operator in terms of a "fluctuation" of the vacuum at hand. For this let us call to mind the definition of the latter (cf. [8]).

Let $(\Theta, \mathcal{V})$ be a vacuum pair that spontaneously breaks a Yang–Mills–Higgs gauge theory that is defined by the data $(\mathcal{P}(\mathcal{M}, G), \rho_{\mathrm{H}}, V_{\mathrm{H}})$. We call a one-parameter family of Yang–Mills–Higgs pairs $(A_t, \varphi_t) \in \mathcal{A}(\xi_{\mathrm{H}}) \times \Gamma(\xi_{\mathrm{H}})$ $(0 \le t \le 1)$ a *fluctuation of the vacuum* if there is a Yang–Mills–Higgs pair (A, φ), such that $A_t = \Theta + t(A - \Theta)$ and $\varphi_t = \mathcal{V} + t\varphi$, where A is supposed to be associated to a non-reducible connection on $\mathcal{P}(\mathcal{M}, G)$ and φ is supposed to be in the "unitary gauge", i.e. $\iota^*\varphi \in \Gamma(\xi_{\mathrm{H,phys}})$. Here, we make use of the fact that the reduced Higgs bundle, when considered as a real vector bundle, decomposes into the Whitney sum

$$\xi_{\mathrm{H,red}} = \xi_G \oplus \xi_{\mathrm{H,phys}} \tag{17}$$

of two real sub-vector bundles representing the Goldstone and the physical Higgs boson (cf. loc sit).

By identifying a connection with its connection form we may write a fluctuation of the canonical Dirac–Yukawa operator $\not{\partial}_{\mathcal{D}}$ as follows:

$$D_{Y,t} =: \not{\partial}_{\mathcal{D}} + t\not{A}_{\mathrm{fl}}, \tag{18}$$

where the "fluctuation" reads: $\not{A}_{\mathrm{fl}} =: \gamma(A - \Theta) + \gamma_5 \otimes G_{\mathrm{Y}}(\varphi)$. We stress that the zero order operator $D_{\mathrm{Y}} - \not{\partial}_{\mathcal{D}}$ defines a fluctuation of the vacuum iff the unitary gauge exists. This holds true, e.g., in the case of rotationally symmetric Higgs potentials, like in the (minimal) Standard Model (see again loc sit). Therefore, every Dirac–Yukawa operator on the fermion bundle can be regarded as a fluctuation of the canonical Dirac–Yukawa operator on the reduced fermion bundle, provided that the unitary gauge exists. Notice that either of the two terms on the right-hand side of (18) transform gauge covariantly with respect to the invariance group of the vacuum. The sum of both, however, is covariant with respect to the original gauge group $\mathcal{G}$. In other words: the fluctuation $\not{A}_{\mathrm{fl}}$ of the vacuum makes the canonical Dirac–Yukawa operator $\not{\partial}_{\mathcal{D}}$ on the (reduced) fermion bundle also $\mathcal{G}$-covariant.

Since the Dirac–Yukawa operator

$$\mathrm{i}\not{\partial}_{\mathcal{D}} \equiv \mathrm{i}\not{\partial} - m_{\mathrm{F}} \tag{19}$$

is the geometrical analogue of Dirac's original first-order operator $\mathrm{i}\not{\partial} - m$, the fluctuation (18) might be regarded as a geometrical variant of what is usually referred to as "minimal coupling". In the case at hand, however, the minimal coupling (18) naturally includes the gravitational field and the states of the Higgs boson. We stress that on the basis of general relativity (without an a priori cosmological constant) it would be inconsistent to assume a non-trivial fermionic mass matrix together with a trivial gravitational field.[6] Thus, a non-trivial ground state of the Higgs boson yields a non-trivial gravitational field, in general.

In this section we have presented a physically intuitive interpretation of a geometrically distinguished class of Dirac-type operators.[7] This interpretation in turn permits a different geometrical interpretation of minimal coupling with the basic feature of including the Higgs boson and the gravitational field. From a geometrical perspective the back and forth of both interpretations may be most transparently summarized by the canonical isomorphism between the fermion bundle and the reduced fermion bundle and the fact that the latter decomposes into the eigenbundles of the fermionic mass matrix. Of course, these identifications depend on the vacuum (pair) only up to gauge equivalence.

So far the three datasets $(\mathcal{M}, g_{\mathrm{M}})$, $(\mathcal{P}(\mathcal{M}, G), \rho_{\mathrm{H}}, V_{\mathrm{H}})$ and $(\mathcal{P}(\mathcal{M}, G), \rho_{\mathrm{F}}, \mathcal{S})$ have been assumed to be given. Moreover, these sets are connected only by a Yukawa mapping (8). In our final section we want to indicate how to connect these sets by postulating a "universal Lagrangian density" that is naturally defined on $\mathcal{D}(\xi_{\mathrm{F}})$.

5. Dirac potentials and Lagrangians

In this paper we consider a fermion as a geometrical object that is defined by the data $(\mathcal{P}(\mathcal{M}, G), \rho_{\mathrm{F}}, \mathcal{S})$, where $(\mathcal{M}, g_{\mathrm{M}})$ is supposed to be given. With respect to this setup there is a distinguished class of first-order differential operators acting on the states of the fermion. As an additional input we considered the data $(\mathcal{P}(\mathcal{M}, G), \rho_{\mathrm{H}}, V_{\mathrm{H}})$ that geometrically defines a Yang–Mills–Higgs gauge theory. In order to combine both datasets we introduced the Yukawa mapping and thereby a specific class of Dirac operators of simple type called general Dirac–Yukawa operators. In fact, the Yukawa mapping generalizes what is known in physics as "Yukawa coupling". If the Yang–Mills–Higgs gauge theory is spontaneously broken, then the fermion decomposes into almost free fermions. Each of these fermions is geometrically represented by a non-flat Clifford module bundle, where the curvature is determined by the gravitational field together with the mass of the free fermion in question. However, since the (pseudo) metric structure has been fixed right from the beginning, these two contributions to the total curvature of the fermion bundle are thus far independent of each other. Of course, when seen from a physical perspective, this seems unsatisfying. One may expect that the masses of the fermions give a contribution to the gravitational field. The most natural way to achieve this is the following construction, which naturally incorporates the dynamics of the gravitational field in the geometrical picture presented here. For this we introduce the following *universal Lagrangian mapping*:

$$\mathcal{L} : \mathcal{D}(\xi_{\mathrm{F}}) \to \Omega^{2n}(\mathcal{M}), \qquad D \mapsto *\mathrm{tr}\, V_{\mathrm{D}}. \tag{20}$$

We call the zero-order operator $V_{\mathrm{D}} =: D^2 - \Delta_{\mathrm{D}} \in \Omega^0(\mathcal{M}, \mathrm{End}(\mathcal{E}))$ the *Dirac potential* associated with D. It is fully determined by the Dirac-type operator in question and can explicitly

[6] Of course, this should not be confounded with the common assumption, e.g., in particle physics, of a *negligible* contribution of the gravitational field.

[7] For instance, Dirac operators of simple type are fully characterized by their Bochner–Lichnerowicz–Weitzenböck decomposition, see, e.g. [1]. Moreover, as already mentioned they constitute the biggest class of Dirac-type operators such that their connection classes have a canonical representative.

be calculated, for instance, by using the generalized Bochner–Lichnerowicz–Weitzenböck decomposition formula (see Eq. (3.13) in [1]). We mention that the Dirac potential generalizes the Higgs potential, for it can be shown that (at least with respect to the Euclidean signature) the Lagrangian of a Yang–Mills–Higgs gauge theory can be recovered from an appropriate Dirac-type operator (see [6]). In this case, however, one has to take into account anti-particles as well (see [7]).

In the case of the canonical Dirac–Yukawa operator $\not{\partial}_{\mathcal{D}}$ one obtains the following Lagrangian density:

$$\mathcal{L}(\not{\partial}_{\mathcal{D}}) = *\mathrm{tr}(\tfrac{1}{4} r_{\mathrm{M}} + m_{\mathrm{F}}^2). \tag{21}$$

Here, $r_{\mathrm{M}} \in \mathcal{C}^{\infty}(\mathcal{M})$ is the scalar curvature with respect to an appropriate $\mathrm{O}(2n)$-reduction of the (oriented) frame bundle of $\mathcal{M}$. As a consequence, space–time must be an Einstein manifold, where the gravitational field is now dynamically determined by the masses of the almost free fermions. Moreover, the mass of an almost free fermion also determines its curvature. In other words: in the ground state both the geometry of space–time and of the (reduced) fermion bundle is determined by the fermionic masses. We summarize this by saying that the "fermionic vacuum" gives rise to a Lagrangian of the form

$$\mathcal{L}(\not{\partial}_{\mathcal{D}}) \sim \langle m_{\mathrm{F}}^2 \rangle \mu_{\mathrm{M}} \tag{22}$$

with $\langle m_{\mathrm{F}}^2 \rangle =: (1/N_{\mathrm{F}}) \sum_{k=1}^{N_{\mathrm{F}}} m_k^2$ and μ_{M} the appropriate volume form determined by the fermionic mass.

Of course, since the Lagrangian (22) is fully determined by the spectrum of the fermionc mass matrix, it is invariant with respect to both the gauge group of general relativity (i.e. the group of volume preserving automorphisms of the oriented frame bundle) and to the invariance group of the vacuum. Moreover, it only depends on the orbit of some minimum and not of the vacua with respect to this minimum. Therefore, the Lagrangian of the fermionic vacuum is indeed a spectral invariant.

6. Summary and outlook

We have presented a geometrical setup permitting a geometrical interpretation of the fermionic masses. Moreover, we have shown how the fermionic mass determines the geometry of space–time and that of the Clifford module bundle which geometrically represents almost free fermions. The fact that the spectral invariant (22) of the fermionic vacuum is proportional to the mean value of the fermionic masses is clearly due to the circumstance that the fermion bundle breaks into a Whitney sum with respect to any given vacuum. This splitting geometrically describes what is usually referred to as "particle multiplet". With respect to a "linear fluctuation" of a vacuum (pair) the canonical Dirac–Yukawa operator becomes covariant with respect to the full gauge group. However, in this case the corresponding canonical Lagrangian determines neither the dynamics of the Higgs boson, nor that of the gauge boson. Moreover, the appropriate Lagrangian reduces to that of the fermionic vacuum. For this it might be more natural to consider a "quadratic fluctuation" of a vacuum. As we have mentioned before, in this case the Lagrangian mapping yields (up to a constant) the bosonic Lagrangian of the Standard Model with gravity included. Notice that for $\dim(\mathcal{M}) = 4$ there are no "higher fluctuations" of the vacuum. Moreover, by geometrically incorporating the notion of anti-particles (i.e. a real structure) the quadratic fluctuations give rise to the same dynamics for the fermions as the linear fluctuations of the vacuum do. This has been discussed, e.g, in [7] for the case of the Euclidean signature. The main reason for using the Euclidean signature was that we are dealing with a universal action instead of a Lagrangian. However, when gravity is taken into account, the latter seems more appropriate since a Lagrangian is a density and thus a purely local object. And because it is a density, the signature of g_{M} does not matter. Moreover, Clifford module bundles always refer to some Clifford bundle over $\mathcal{M}$. But this in turn obviously refers to some chosen $\mathrm{O}(2n)$-reduction of the frame bundle of $\mathcal{M}$, i.e. to some fixed g_{M}. However, when g_{M} is physically interpreted as a gravitational field, it cannot be fixed a priori, for it has to satisfy, e.g., Einstein's equation. This is obviously a dilemma one always has to face in if gravity is taken into account. The philosophy of the paper at hand with respect to the Lagrangian mapping (20) is as follows: the field equations determined by the corresponding Lagrangian are considered as "constraints" of how to glue together the local pieces to give rise to global geometrical objects like, e.g., the fermion bundle. We consider this interpretation of the Euler–Lagrange equations to hold true, especially in the case of the Einstein equation.

Interestingly, there are certain parallels between the geometrical setup presented here and what is called "almost commutative models" in the literature. In particular, the canonical Dirac–Yukawa operator corresponds to the "total Dirac operator" and $\mathcal{D}$ to the "internal Dirac operator" in the Connes–Lott description of the Standard Model within the frame of A. Connes' non-commutative geometry (see, e.g. [3] or [4]). Like in the case of the Connes–Lott model one also has a "fermion doubling" in the geometrical frame presented here. This still has to be carefully analyzed, for we can perhaps work with the physical signature of g_{M}. Concerning quantization it seems challenging to try to understand what it geometrically means to "quantize" the above-mentioned constraints. This, of course, is still an open question and has not been addressed in this paper. Instead, the main objective here was to explore the geometrical meaning of the fermionic "mass without mass".

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